Carnot Engine Operating Upon Exhaustible Thermal Potential with Respect to Heat

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(Dated: February 19, 2015)

A Carnot engine bridges a thermal potential to perform work. Since one or both reservoirs are exhaustible, the potential is nonrenewable. A formula for the progression of cumulative work performed as well as instantaneous efficiency rate as a function of cumulative heat flow is derived, and an example is presented. A simulator in the Ruby programming language is utilized to plot these relationships.

I. INTRODUCTION

A Carnot Engine extracts work W from a thermodynamic potential comprising a thermal difference, typically the result of two heat reservoirs of differing temperatures.[1] The hot reservoir is often exhaustible. If so, even if the cooler reservoir represents an inexhaustible reservoir, nevertheless, this system comprises an exhaustible thermal potential.

A Carnot engine operating over an exhaustible thermal potential is similar to a hot object emitting heat in a cold environment, but with two important differences. First, the rate of heat flow will be governed by the characteristics of the engine rather than the temperature difference between the reservoirs. Second, an engine extracts work from the thermal difference, but the Second Law of Thermodynamics requires W to be less than the flow of heat Q_h from the hot reservoir. The is accomplished by discounting the transfer of heat by an efficiency multiplier ϵ. This paper will express the consumption of the thermal potential as a function of cumulative heat flow out of an exhaustible hot reservoir.

A. The Carnot Engine

It will be useful to review the classic Carnot engine example commonly found in physics textbooks, and then delve slightly further into that example. Typically, a Carnot Engine (Figure 1) is shown to operate between two thermal reservoirs. One reservoir is is designated as "hot" with temperature T_h while the other is designated as "cold" with temperature T_c. Heat Q_h flows out of the hot reservoir into the engine, part of which is converted into work W, while the remainder Q_c is exhausted into the cold reservoir. (Let it be clear that in this article, Q_h represents cumulative heat flow, rather than the instantaneous rate of heat flow.)

The proportion of the energy Q_h flowing out of the hot reservoir converted into work W is the efficiency ϵ of that engine, known as its Carnot efficiency. The formula for the efficiency of a Carnot engine is:

$$\epsilon = 1 - \frac{T_c}{T_h}. \quad (1)$$

The engine must have a positive efficiency to perform

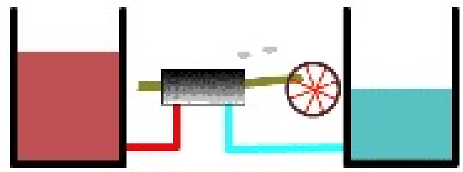

FIG. 1. A Carnot Engine pulls heat from the hot reservoir (left), performs work to turn the wheel (center) and exhausts waste heat to the cold reservoir (right).

work. This only occurs when T_h exceeds T_c. Temperature is typically expressed in a unit of absolute temperature such as Kelvin, denoted by the letter K. Work W is

$$W = \epsilon Q_h. \quad (2)$$

For a more detailed explanation of the functioning of a Carnot engine, including the Carnot cycle, see Stowe (1984).[2]

B. Examining the Reservoirs

In the classic case, the reservoirs are inexhaustible.[3] One can remove an infinite amount of heat from the hot reservoir and place the same into the cold reservoir without any change in the temperature of either reservoir. Often, this is not a bad approximation of real life, such as when the temperature of the hot reservoir is precisely controlled by adjusting the burn rate of fuel and the cold reservoir is a large, well circulating body of water. In less ideal cases, the characteristics of the reservoirs that give rise to temperature become important.

The temperature of a reservoir depends upon its thermal energy U, volume V, and the heat capacity per unit volume C.

$$T = \frac{U}{CV}. \quad (3)$$

Other factors such pressure and the characteristics of the reservoir substance can also play a role, but a discussion of such is beyond the scope of this paper.

II. ONE EXHAUSTIBLE RESERVOIR

We will consider situations where the hot reservoir is exhaustible, while the cold reservoir is inexhaustible. Although the following approach can be applied two systems with two exhaustible reservoirs, the principles are the same as with one.

A. Amount of Available Energy

We will simplify matters by assuming that the reservoirs comprise a single type of material that does not undergo phase changes (such as vaporization or freezing). Although, heat capacity C can change depending on pressure, we will assume a constant C for our substance and constant reservoir volume V. Then temperature T_h depends only upon the reservoir's thermal energy U_h.

$$T_h = \frac{U_h}{CV}. \tag{4}$$

The initial temperature of the hot reservoir shall be T_{h0}, while its initial energy shall be U_{h0}. The temperature of the cold reservoir shall remain fixed at T_c. As Q_h flows, U_h shall decrease, and T_h will consequently diminish from its initial temperature temperature to become equal to that of the environment (T_c).

How much energy U_h must be removed from the hot reservoir to bring T_h down to T_c, if T_h is originally greater than T_c? The total amount of energy that can be removed is the difference between the initial hot temperature and the constant cold temperature of the cold reservoir, multiple by the volume and capacity of the hot reservoir.

$$U_{h_{T_h}} - U_{h_{T_c}} = T_h C_h V_h - T_c C_h V_h.$$

Simplified,

$$U_{h_{T_h}} - U_{h_{T_c}} = (T_h - T_c) C_h V_h. \tag{5}$$

The above amount can be considered as potential Q_h. This is the maximum amount of heat that would spontaneously flow from the hot reservoir into the cold one if they were in direct thermal contact:

$$Q_{h_{max}} = (T_h - T_c) C_h V_h. \tag{6}$$

How how much U_h is needed to raise T_h from zero to T_c? We begin with Equation (3):

$$T = \frac{U}{CV}.$$

Rearranging the terms produces:

$$U = TCV.$$

$$U_{h(0 \to T_h)} = 0 + T_c C_h V_h.$$

$$U_{h(0 \to T_h)} = T_c C_h V_h. \tag{7}$$

This is an interesting question, because it represents the amount of thermal energy in the hot reservoir when it is in equilibrium with the cold reservoir. When the two reservoirs are in equilibrium, none of this energy is available to perform work.

B. Efficiency

Let us first consider the response of efficiency ϵ to cumulative heat flow Q_h. Since ϵ depends on T_h, which in turn depends upon U_h, Equation 1 becomes

$$\epsilon = 1 - \frac{T_c}{(\frac{U_h}{VC})}.$$

Since U_h decreases as Q_h increases,

$$\epsilon = 1 - \frac{T_c}{(\frac{U_{h_0} - Q_h}{VC})},$$

and rearranging,

$$\epsilon = 1 - \frac{VCT_c}{U_{h_0} - Q_h}. \tag{8}$$

C. Rate of Work as a Function of Cumulative Heat Flow Q_h

Now let us examine the rate of work W that this system can perform as a function of cumulative heat flow Q_h. Although work W is

$$W = \epsilon Q_h, \tag{9}$$

efficiency ϵ is not a constant when a reservoir is exhaustible. So we obtain a differential equation:

$$dW = (1 - \frac{VCT_c}{U_{h_0} - Q_h}) dQ_h.$$

D. Total Work Performed as a Function of Cumulative Heat Flow Q_h

Since we are considering work W performed over a period of time, we need to integrate the above expression. After integrating, we get:

$$W = Q_h + VCT_c ln(U_{h_0} - Q_h) + k,$$

where k is a constant of integration. We need to determine k. First, we set Q_h to 0, so that

$$W = 0 + VCT_c ln(U_{h_0}) + k.$$

Since work $W = 0$ at $Q_{h_0} = 0$:

$$0 = VCT_c ln(U_{h_0}) + k.$$

Consequently,

$$k = -VCT_c ln(U_{h_0}).$$

So substituting the above expression for k, our full expression for work W with respect to cumulative heat flow Q_h is then:

$$W = Q_h + VCT_c ln(U_{h_0} - Q_h) - VCT_c ln(U_{h_0}).$$

Simplifying,

$$W = Q_h + VCT_c(ln(U_{h_0} - Q_h) - ln(U_{h_0})).$$

$$W = Q_h + VCT_c ln(\frac{U_{h_0} - Q_h}{U_{h_0}}).$$

$$W = Q_h + VCT_c ln(1 - \frac{Q_h}{U_{h_0}}). \quad (10)$$

E. Maximum Possible Work

The available work of this system that can be performed before T_h falls to T_c is also the point at which no more heat Q_h will flow. Since we know the maximum amount of heat that can flow, we can deduce the maximum possible work W_{max} of the system. Q_{max} is:

$$Q_{h_{max}} = (T_h - T_c)C_h V_h,$$

Recall that Equation 10 for work W is

$$W = Q_h + V_h C_h T_c ln(1 - \frac{Q_h}{U_{h_0}}).$$

Then the maximam possible work W_{max} is

$$W_{max} = Q_{h_{max}} + V_h C_h T_c ln(1 - \frac{Q_{h_{max}}}{U_{h_0}}).$$

Simplifying,

$$W_{max} = (T_h - T_c)C_h V_h + V_h C_h T_c ln(1 - \frac{(T_h - T_c)C_h V_h}{U_{h_0}}),$$

we finally obtain the following expression for W_{max}:

$$W_{max} = C_h V_h \left(T_h - T_c + T_c ln \left(1 - \frac{(T_h - T_c)C_h V_h}{U_{h_0}} \right) \right). \quad (11)$$

F. Special Case

What about the case when $T_h = T_c$? We expect that $W = 0$, since combining Equation 1 and Equation 2 produces

$$W = Q_h(1 - \frac{T_c}{T_h}),$$

and when $T_h = T_c$, then $T_h/T_c = 1$, so that:

$$W = Q_h(1 - 1) = 0.$$

G. Example: calculated results from one exhaustible heat reservoir

Here, a Carnot engine operates across an exhaustible reservoir at initial temperature of 1000 K, with volume = 1 m^3 and heat capacity of 1 J/K, and an inexhaustible reservoir at fixed temperature of 300 K. Table I shows the results from our formula. We allow the heat to flow in steady, constant increments. The energy of the hot reservoir U_h consequently decreases in similar increments. Efficiency ϵ drops slowly, them more quickly. The pace of work performed ΔW starts briskly, then increasingly slows.

Q_h (J)	U (J)	W (J)	ϵ (%)*	ΔW
0	1,000,000	0	70.00	0
100,000	900,000	68,392	68.333	68,392
200,000	800,000	133,057	64.583	64,665
300,000	700,000	192,998	59.821	59,941
400,000	600,000	246,752	53.571	53,755
500,000	500,000	292,056	45,000	45,304
600,000	400,000	325,113	32.500	33,057
700,000	300,000	338,808	12.500	13,695

TABLE I. Results using formula. *Approximate mean efficiency for most recent increment of heat flow.

III. RUNNING THE SIMULATION

A simulator has been written in the Ruby programming language. Let us compare the simulator with results from our equation. The simulator iteratively executes a loop, with the results from each increment drawing from the previous increment's results.

As before, a Carnot engine operates across an exhaustible reservoir at initial hot reservoir temperature T_h of 1000 K, with volume = 1 m^3, a heat capacity of 1 J/K, and an inexhaustible reservoir at fixed temperature of 300 K.

When running the simulation, we will assume no phase changes. We will assume that the volume of the materials in the heat reservoirs stays constant with the relevant temperature changes, so that volume is strictly proportional to mass (for purposes of the specific heat and heat capacity.

A plot of the simulation run and the results are shown graphically. Figure 2 shows efficiency as a function of Q_h. Note that the initial efficiency is 70%, which is determined solely by the initial reservoir temperatures. At first, the efficiency decays gradually, then more quickly. Yet at its point of fastest decay, efficiency suddenly reaches zero since the thermal potential has been completely consumed by this point.

Figure 3 shows the declining rate of work due to decreasing efficiency. That this plot is very similar in appearance to that shown in Figure 2 is not surprising, since work W is a strict function of efficiency ϵ given the constraints of the example simulated. If this system allowed efficiency to become negative, then negative work would be performed, and this system would become a refrigerator. This would only be possible if work was obtained from outside of this system or as a stored on-thermal source of potential energy.

Figure 4 shows cumulative work as a function of cumulative heat flow. The heat stops flowing at the end of the graph, so the work shown at that point is the maximum potential work that can be extracted from this system.

See the simulator program code at: `https://github.com/mciotola/carnot_exhaustible_reservoir`.

IV. CONCLUSIONS

Both the formula and simulation show that for an exhaustible reservoir, efficiency and the rate of work production decrease first slowly then more quickly as a function of cumulative heat transfer.

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[1] Daniel V. Schroeder. *An Introduction to Thermal Physics.* Addison-Wesley, 2000.

[2] Keith Stowe. *Introduction to Statistical Mechanics and Thermodynamics.* John Wiley and Sons, New York, 1984.

[3] Mark P. A. Ciotola. Simulator for the efficiency and work of carnot engines. *HeatSuite Journal*, 1(3), 12 November 2014.

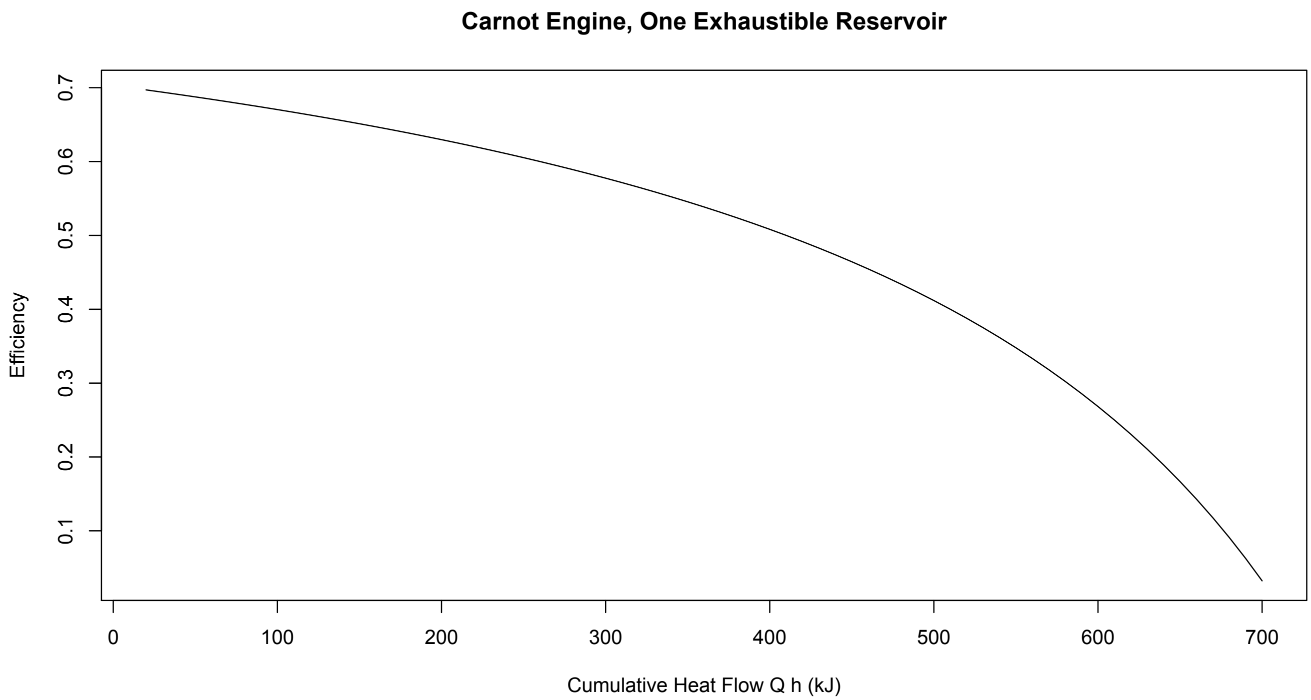

FIG. 2. Efficiency versus heat flow

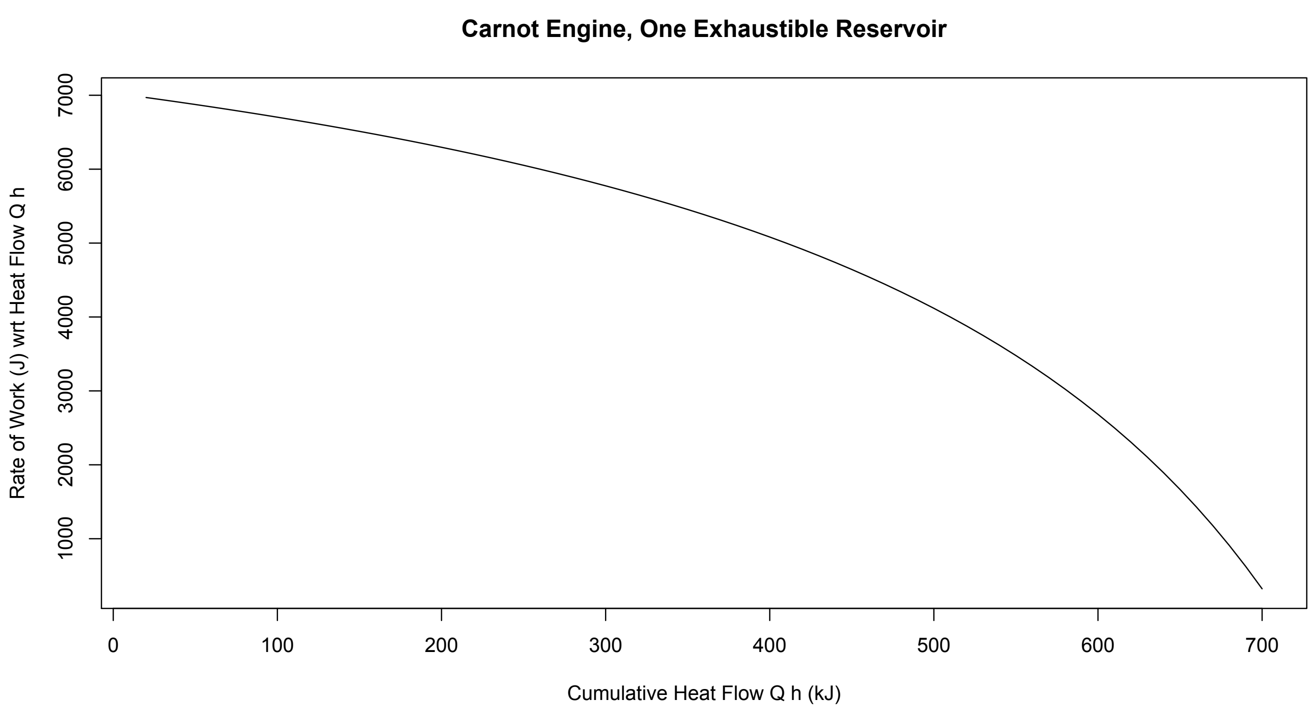

FIG. 3. Rate of work versus heat flow

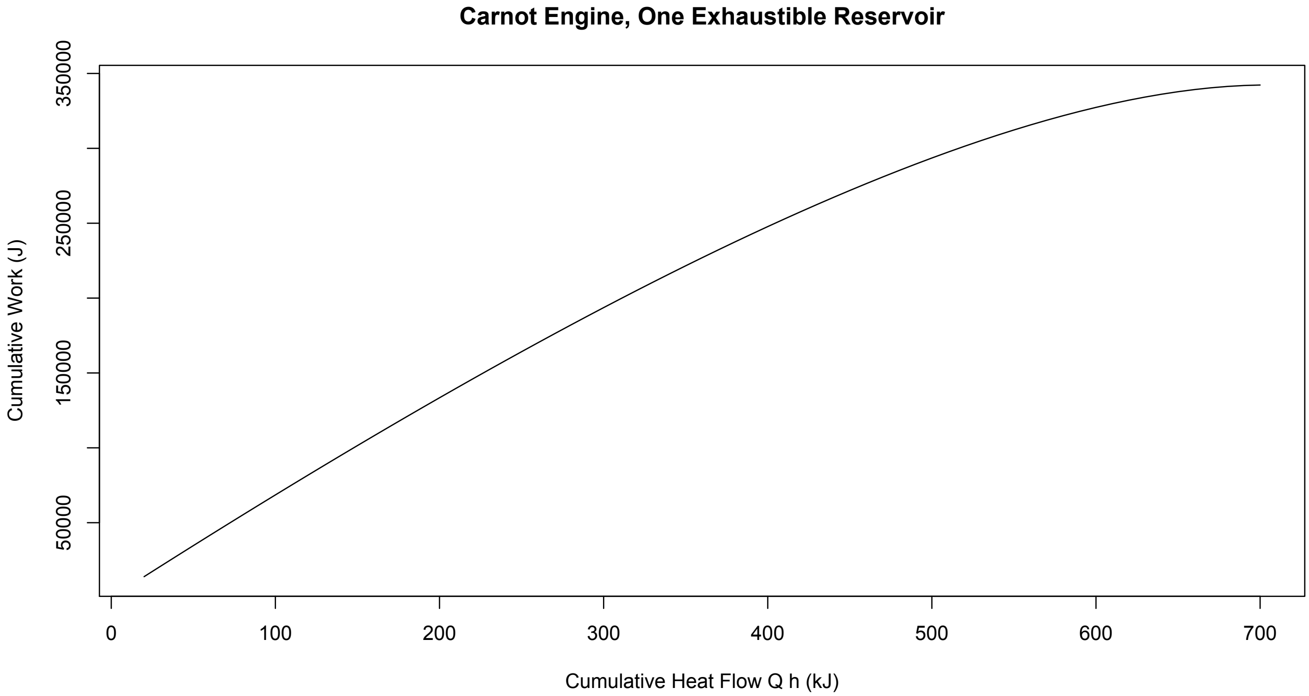

FIG. 4. Cumulative of work versus heat flow