Ruby Simulation of Newton's Law of Cooling

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Fourier's Law is introduced. Fourier's Law is shown as a prelude to Newton's Law of Cooling, then contrasted with that law. Newton's Law of Cooling derived and demonstrated with an example. A simulation produces additional results which show rapid temperature decline and then a long "tail".

I. INTRODUCTION

Fourier's Law expresses the flow of heat through a conductor bridging thermal conductors of differing temperatures. Fourier's Law is typically presented in textbooks involving a thermal conductor bridging a hot and cold reservoir, where the temperature of each reservoir is considered to be fixed.[1] (Strictly speaking the term *heat* only applies to where thermal energy flows, so the term *heat flow* is redundant. Yet redundancy adds emphasis!) Heat Q flows through the conductor, and the rate of that flow is dependent upon both conductor characteristics and the temperature difference ΔT between the reservoirs. Fourier's Law can be expressed as:

$$\frac{dQ}{dt} = \frac{kA\Delta T}{L} \tag{1}$$

where t is time, k is a constant dependent upon the material comprising the conductor, A is conductor area and L is conductor length. Since neither the conductor characteristics nor the reservoir temperatures change in the classic scenario, the heat flow remains constant.[2]

In contrast, Newton's Law of Cooling concerns a hot object surrounded by a cold environment, as shown in Figure 1. (The terms *hot* and *cold* are only meant relative to each other and not in any absolute sense). A warmer object immersed in a cooler environment will eventually cool down to the temperature of the environment, but the rate of cooling is nonlinear. See Winterton (1999) for a history and detailed discussion of Newton's Law of Cooling.[3]

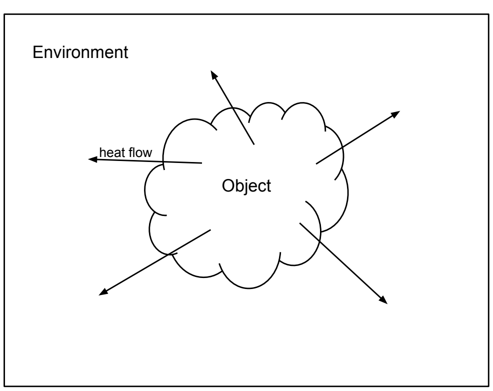

FIG. 1. Newton's Cooling Law

II. DERIVATION OF NEWTON'S LAW OF COOLING

A hot object initially at temperature T_0 cools down in an environment with a constant temperature of T_e. We will assume an environment large enough to absorb heat from the object without itself changing temperature. The object involved is further assumed not to undergo phase changes in the range of temperatures considered. The temperature of the object at any point of time is T_t. As the object cools down, it will eventually reach equilibrium with its environment so that $T_t = T_e$.

We have the following expectations for this law. At t = 0, T_t will equal T_0, by definition. At t = infinity, T_t = T_e, which is another way to say that the object will eventually achieve thermal equilibrium with its environment, so that any thermal energy flowing into the object will equal that flowing out.

We can utilize Fourier's Law to derive Newton's law of Cooling. If the object is considered the hot reservoir and the environment the cold reservoir, and the characteristics of the object remain constant over time, then we can expect heat flow to be proportional to temperature difference T_t - T_e at any point of time. Since the rate of change is negative, we use a - sign in front of K_1. So we expect:

$$\frac{dQ}{dt} = -k_1(T_t - T_e), \tag{2}$$

where k_1 is a constant that is dependent upon the characteristics of the object such as its material, volume and shape.

However, T_t is not constant. As the object cools, its temperature decreases, so that the temperature difference decreases and the rate of heat flow decreases. So dQ/dt is proportional to T_t. Although, for Newton's Law of Cooling, we wish to be able to derive the law without direct reference to heat flow, this observation brings us closer to the actual law.

So we need to express T_t as a function of time (or at least as a function of cumulative heat flow). Since the conductor characteristics remain constant, dT_t is proportional to dQ, so we utilize a new constant k_2 which incorporates the old constant k_1.

$$\frac{dT_t}{dt} = -k_2(T_t - T_e). \tag{3}$$

Next, we rearrange the terms:

$$dT_t = -k_2(T_t - T_e)dt.$$

$$\frac{dT_t}{(T_t - T_e)} = -k_2 dt.$$

Next, we integrate both sides:

$$\int \frac{dT_t}{(T_t - T_e)} = \int -k_2 dt,$$

and get:

$$ln(T_t - T_e) = -k_2 t + C_1.$$

We raise each side to e:

$$(T_t - T_e) = -e^{k_2 t + C_1}.$$

We again rearrange the terms:

$$T_t = T_e - e^{k_2 t + C_2},$$

where

$$T_t = T_e - e^{C_2} e^{k_2 t}.$$

If we set t to 0 (the initial condition),

$$T_t = T_0 = T_e - e^{C_2},$$

then

$$e^{C_2} = (T_e - T_0).$$

Hence,

$$T_t = T_e - (T_e - T_0)e^{k_2 t}.$$

Rearranging, and denoting k_2 simply as k, we finally get

$$T_t = T_e + (T_0 - T_e)e^{-kt}, \qquad (4)$$

which is Newton's Law of Cooling. (Also see the related UCB page).[4]

III. AN EXAMPLE

If the object with a temperature is 500 K is in an environment with a temperature of 300 K, and $k = 0.25s^{-1}$, what will be the object's temperature after 5 seconds, 10 seconds, 15 seconds and 20 seconds? The cooling function will be as follows:

$$T_t = 300\ K + (500\ K - 300\ K)e^{-0.25s^{-1}t}. \qquad (5)$$

A calculator or MS Excel can be used to plug time into the formula and calculate the temperature. The results are shown in Table I.

As can be seen, the decrease in object temperature is much greater in the first five seconds than the last five seconds. After 20 seconds, T_t has nearly reached environmental temperature T_e.

IV. SIMULATOR

A simulator has been created in the Ruby programming language, and can be found at `<https://github.com/mciotola/newtons_law_of_cooling_analytical>`

Figure 2 shows the simulator results for object temperature T_t versus time t. The simulation results plotted in Figure 2 are consistent with those shown in Table I. There is a rapid fall in object temperature, but the rate of decrease slows, producing a long "tail" before T_t falls to T_e. Mathematically, the tail is infinitely long, but in real life, the temperature will fall to within the experimental uncertainties in the data, and the system can be considered at equilibrium at that point.

V. CONCLUSION

Newton's Law of Cooling is an example of exponential decay. This sort of decay applies to a wide variety of cases. It occurs when the rate of something is proportional to the negative value of itself. An interesting variation is when a cold object is placed in a hot environment. Newton's Law of Cooling will still give the correct result, except that the object's temperature will increase with time until it reaches that of its environment.

An interesting aspect of Newton's Law of Cooling is that the rate of temperature change is in no way dependent upon absolute temperature, but rather the difference in temperatures. A 100 K difference at 300 K versus 500 K will produce the same heat flow as a 800 K versus 1000 K difference. Nevertheless, it should be noted that many physical laws are reasonably accurate for only a limited range of values. So while Newton's Law might be suitably accurate for the sort of temperatures encountered in everyday life, it might not provide the required level of accuracy for very high or low temperatures.

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[1] Daniel V. Schroeder. *An Introduction to Thermal Physics.* Addison-Wesley, 2000.

Time t (s)	Environment Temperature T_e (K)	Object Temperature T_t (K)
0	300	500
5	300	357
10	300	316
15	300	305
20	300	301

TABLE I. Object Temperature versus Time

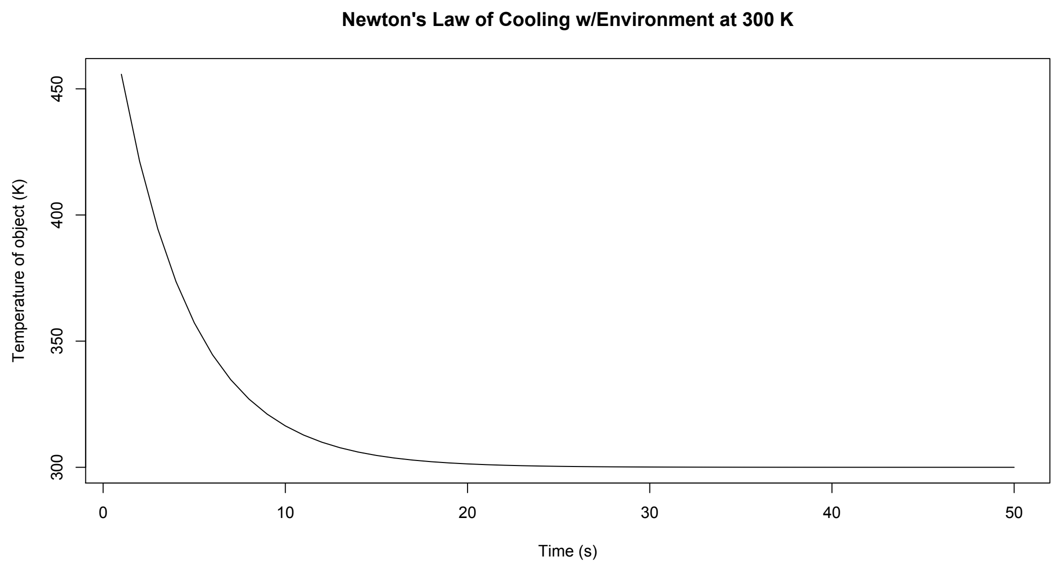

FIG. 2. Newton's Cooling Law

[2] Mark P. A. Ciotola. Fourier's law of conduction. *HeatSuite Journal*, 26 February 2014.
[3] R. H. S. Winterton. Newton's law of cooling. *Contemporary Physics*, 40(3):205–212, 08 Nov 2010.
[4] University of British Columbia. Ubc calculus online course notes: Other differential equations.