A Simulator for the Efficiency and Work of Carnot Engines

Mark Ciotola

November 12, 2014

Abstract

The Carnot heat engine is reviewed with examples. A Carnot engine simulator in the Ruby programming language and a web app written in Ruby on Rails are discussed.

1 Introduction

Heat engines bridge thermal reservoirs of differing temperatures in order to transform heat flow into work, such as powering a pump, train or electric generator. Early engine developers strove to make their engines increasingly efficient. Eventually, they wondered what limits may exist on heat engine efficiency, if any. Sadi Carnot, a French military officer, attempted to answer this question by conceiving a heat engine with the greatest hypothetical efficiency. The resulting Carnot engine is a hypothetical heat engine that produces the maximum amount of work without producing any net entropy (e.g. Schroeder 2000). Hence, it is a reversible process.

2 The Carnot Engine

2.1 Heat flow and work

Typically, a Carnot Engine is shown to operate between two thermal reservoirs (see Figure 1). The warmer reservoir is designated as "hot" with temperature T_h while the cooler one is designated as "cold" with temperature T_c. Heat Q_h flows out of the hot reservoir into the engine, which transforms part of that energy into work W, while the engine exhausts the remaining heat Q_c into the cold reservoir. Since the First Law of Thermodynamics states that energy must be conserved, then

$$Q_h = W + Q_c. \tag{1}$$

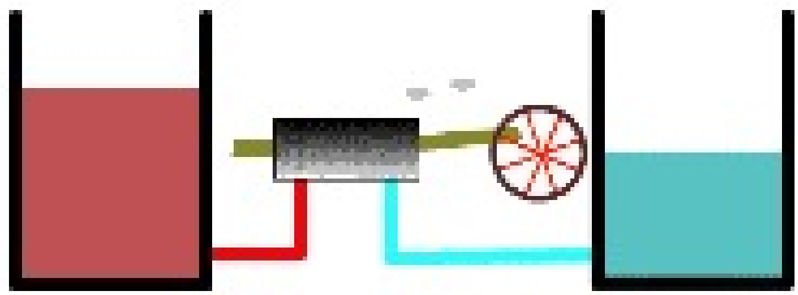

Figure 1: A Carnot engine bridges a hot reservoir (left) and a cold reservoir(right)

2.2 Entropy and efficiency

The proportion of the energy Q_h flowing out of the hot reservoir that is transformed into work W is the efficiency ϵ of that engine. This is true of any engine. So

$$\epsilon = \frac{W}{Q_h}. \tag{2}$$

However, we can be more specific for a Carnot engine. According to Second Law of Thermodynamics, a quantity called *entropy* cannot decrease for an isolated system. The efficiency ϵ for a Carnot engine is the highest possible efficiency that does not violate that law. Any real life engine could not operate at a higher efficiency, and in many cases will do much worse. Therefore, a Carnot engine neither increases nor decreases total entropy S:

$$S_{c_{increase}} - S_{h_{decrease}} = 0. \tag{3}$$

For a Carnot engine, any entropy decrease caused by removing Q_h from the hot reservoir must be matched by an equal increase of entropy by exhausting Q_c into the cold reservoir.

$$S_{h_{decrease}} = -S_{c_{increase}}. \tag{4}$$

Decreasing the thermal energy of a reservoir decreases its entropy. Conversely, increasing its energy increases its entropy. Note that a change of thermal reservoir energy U by an amount of heat Q results in an entropy change S of Q/T, rather than just Q.

$$S_{h_{decrease}} = -\frac{Q_h}{T_h}. \tag{5}$$

$$S_{c_{increase}} = \frac{Q_c}{T_c}. \tag{6}$$

Then Equation 4 can be restated as:

$$\frac{Q_c}{T_c} - \frac{Q_h}{T_h} = 0 \tag{7}$$

and as

$$\frac{Q_h}{Q_c} = \frac{T_h}{T_c}. \tag{8}$$

Consequently, a unit of energy removed from the hot reservoir will cause a smaller entropy change than adding a unit of energy to the cold reservoir. So for total entropy to remain constant, the entropy S_c increase for each unit of Q_c *must* exceed the entropy S_h decrease for each unit of Q_h.

$$\frac{dS_c}{dQ_c} > \frac{dS_h}{dQ_h}. \tag{9}$$

Since T_h is greater than T_c, Q_h is likewise greater than Q_c. With this confirmed, we can now determine the efficiency of a Carnot engine. Efficiency means the proportion of Q_h that becomes work W. So

$$\epsilon = \frac{Q_h - Q_c}{Q_h} \tag{10}$$

and

$$\epsilon = 1 - \frac{Q_c}{Q_h}. \tag{11}$$

Using Equation 8,

$$\frac{Q_c}{Q_h} = \frac{T_c}{T_h}, \tag{12}$$

the efficiency of a Carnot engine becomes:

$$\epsilon = 1 - \frac{T_c}{T_h}. \tag{13}$$

It is evident that efficiency of the engine depends upon the temperature difference of the thermal reservoirs, with a slight twist. There has to be some difference in temperature for the engine to have any non-zero efficiency. Yet, regardless of any temperature T_h of the hot reservoir, the efficiency will be 100% only if the temperature of the cold reservoir $T_c$is 0 K. The unit K represents Kelvin, which is an absolute measurement of temperature, so that 0 K is the lowest possible temperature. We will use Kelvin for units of temperature, since an absolute unit of temperature is required for our formulas to provide correct results.

In real life, it is difficult to get T_c to anywhere near 0 K. Real heat engines require a large reservoir which to exhaust heat, and this is typically the Earth's atmosphere or a cooling pond, which is at about 300 K. Since it is typically too expensive to cool the cold reservoir to below the temperature of its environment, real life engines are often made more efficient by designing and operating them to work with a high T_h. Note also that real life heat engines are typically less efficient that Carnot efficiency, which is merely the theoretical maximum efficiency. The difference between Carnot efficiency and actual efficiency is known as Second Law efficiency. Further discussion of such is beyond the scope of this paper.

Now that we know Carnot efficiency, we can determine amount of work a Carnot engine can perform for particular reservoir temperatures:

$$W = \epsilon Q_h = (1 - \frac{T_c}{T_h})Q_h. \tag{14}$$

3 Examples

3.1 A trivial example

Imagine that we tried to operate a Carnot engine using the atmosphere for both the hot and cold reservoirs. Such an engine would not require any fuel such as gasoline or wood so it would be inexpensive to operate. Let us examine this engine. The temperature T_h of the hot reservoir would be 300 K and the temperature T_c of the cold reservoir would be 300 K. Efficiency is then:

$$\epsilon = 1 - \frac{300K}{300K} = 1 - 1 = 0$$

With an efficiency equal to zero, this engine is not capable of producing any work. This example demonstrates the necessity of a temperature difference for a Carnot engine to produce work, regardless of how much heat energy exists in the atmosphere.

3.2 A more meaningful example

This time, the reservoirs will have temperatures of 300 K and 500 K. We allow 10 J of Q_h to flow. J represents a unit of energy called Joules (pronounced *jewels*). Efficiency is:

$$\epsilon = 1 - \frac{300K}{500K} = 1 - 0.6 = 0.4,$$

which is 40%. Then the work produced will be:

$$0.4 * 10J = 4J.$$

Waste heat exhausted into the cold reservoir will be:

$$Q_c = Q_h - W = 10J - 4J = 6J.$$

Entropy decrease of the hot reservoir will be:

$$\Delta S_h = \frac{\Delta Q}{T_h} = 10J/500K = 0.02J/k.$$

Entropy increase of the cold reservoir will be:

$$\Delta S_c = \frac{\Delta Q}{T_c} = 6J/300K = 0.02J/k.$$

The net change of entropy of the system is:

$$\Delta S_c - \Delta S_h = 0.$$

Plugging in our numbers, this is:

$$0.02\frac{J}{K} - 0.02\frac{J}{K} = 0.$$

So the total entropy has not decreased, yet this example is consistent with the Second Law of Thermodynamics.

3.3 Revving up the efficiency!

Let's see what happens when we keep T_c the same, but we greatly increase T_h. Reservoirs of 300K and 1000 K. Q_h will remain at 10 J. Efficiency is:

$$\epsilon = 1 - \frac{300K}{1000K} = 1 - 0.3 = 0.7.$$

Work will be:

$$0.7 * 10J = 7J.$$

So by doubling T_h, we have nearly doubled the efficiency and hence the work performed, all for the same 10 J of heat flow. Waste heat exhausted into the cold reservoir will be:

$$Q_c = Q_h - W = 10J - 7J = 3J.$$

Entropy decrease of the hot reservoir will be:

$$\Delta S_h = \frac{\Delta Q}{T_h} = 10J/1000K = 0.01J/k.$$

Entropy increase of the hot reservoir will be:

$$\Delta S_c = \frac{\Delta Q}{T_c} = 3J/300K = 0.01J/k.$$

The net change of entropy of the system is:

$$0.01\frac{J}{K} - 0.01\frac{J}{K} = 0,$$

as is expected.

4 Online simulation code and ready-to-use web application

4.1 Ruby computer language Carnot engine simulation

An open-source simulation has been written in the Ruby programming language. Ruby is easy to learn well-suited to introductory simulations. It is also well suited to redeploying such simulations as web applications (Ciotola 2013). View or download the code at: `https://github.com/mciotola/carnot_heat_engine`

4.2 Ruby on Rails Carnot engine simulator web application

A Ruby on Rails Carnot engine simulator web application has been set up for readers who wish to use the application immediately without downloading code or setting up a Ruby or command line environment. Ruby on Rails is a *model-view-controller* framework for the Ruby language. Users can specify reservoir temperatures and Q_h. See the simulator at: `http://www.heatsuite.com/?page_id=69`

5 Discussion

The Carnot engine is not practical for actual use, since it must operate extremely slowly. However, the Carnot Engine is a useful educational tool as well as a way to place an upper limit on possible real life heat engine efficiency.

5.1 Delving Deeper: Examining the Reservoirs

Until now, we have assumed that the reservoirs are infinite in volume, so that their temperatures do not change. With this assumption, the reservoirs can be considered inexhaustible. One can remove an infinite amount of hear from the hot reservoir and place the same into the cold reservoir without the temperature of the reservoirs changing. This assumption is not necessary for the operation of a Carnot engine, but simplifies the math. Further, it is not an unreasonable approximation for many physical scenarios, such as when the temperature of the hot reservoir is precisely controlled by adjusting the burn rate of fuel and the cold reservoir is a large, well circulating body of water.

Yet, in less ideal cases, the characteristics of the reservoirs giving rise to temperature become important. Let us study a few such cases. Reservoir temperature T depends upon the thermal energy U in a reservoir divided by its volume V and the heat capacity C of the reservoir substance. For this example, we assume a constant volume V; also that heat capacity C can change depending on pressure, but we assume C for our substance is constant. However, if the thermal energy U changes, the temperature T will proportionally change:

$$\Delta T = \frac{\Delta U}{CV}. \tag{15}$$

To keep things simple, we have assumed that the reservoirs are made of a single type of material that does not undergo phase changes (such as vaporization or freezing). However, is there is a phase change involved, the relation between temperature T and thermal energy U added or removed is no longer simply linear.

An easily visualized case is where the cold reservoir is a vat of ice water. As long as both ice and water remain in the reservoir, T_c will remain at about 273 K (the freezing point of water in typical atmospheric conditions). Only once the ice all melts or the water freezes will T_c change. This is because water must release extra energy to become ice, or absorb extra energy to break apart the bonds between ice molecules and melt. So for a limited range of energy change, there will not be a temperature change.

5.2 Contrast with Fourier's Law

Fourier's Law holds that heat flow in a material bridging a thermal difference is proportional to the magnitude of that difference (e.g. Schroeder 2000; Ciotola 2014). However, for a Carnot engine, heat energy flow Q_h from the hot reservoir is determined arbitrarily. In real life, it will be a characteristic of the engine based upon its design and construction.

References

[1] Ciotola, Mark, "Exponential Growth Simulator Utilizing the Ruby Language", *Heat-Suite Journal* 5 April 2013.

[2] Ciotola, Mark, "Fourier's Law of Conduction", *HeatSuite Journal* 26 February 2014.

[3] Schroeder, Daniel V., *An Introduction to Thermal Physics*, 2000.

A An Alternative Derivation of Carnot Efficiency

Below is an alternative way to derive Carnet efficiency. It is more complicated, but begins with the First Law of Thermodynamics (conservation of energy). First, we find expression for Q_h and Q_c:

$$Q_h = \frac{Q_c T_h}{T_c} \tag{16}$$

and

$$Q_c = \frac{Q_h T_c}{T_h}. \tag{17}$$

Recalling that

$$Q_h = W + Q_c, \tag{18}$$

we plug in our above expressions for Q_h and Q_c:

$$\frac{Q_c T_h}{T_c} = W + \frac{Q_h T_c}{T_h}, \tag{19}$$

so that:

$$W = \frac{Q_c T_h}{T_c} - \frac{Q_h T_c}{T_h}. \tag{20}$$

Further recalling that

$$\epsilon = \frac{W}{Q_h}, \tag{21}$$

we find that

$$\epsilon = \frac{\frac{Q_c T_h}{T_c} - \frac{Q_h T_c}{T_h}}{Q_h}. \tag{22}$$

Once again recalling that

$$Q_h = \frac{Q_c T_h}{T_c}. \tag{23}$$

we replace Q_h to get

$$\epsilon = \frac{\frac{Q_c T_h}{T_c} - \frac{\frac{Q_c T_h}{T_c} T_c}{T_h}}{\frac{Q_c T_h}{T_c}}. \tag{24}$$

Then, factoring out Q_c gets

$$\epsilon = \frac{\frac{T_h}{T_c} - \frac{\frac{T_h}{T_c} T_c}{T_h}}{\frac{T_h}{T_c}}. \tag{25}$$

Eliminating a factor of $1/T_c$ gets

$$\epsilon = \frac{T_h - \frac{(T_h T_c)}{T_h}}{T_h}. \tag{26}$$

Further cleaning up results in a common expression for Carnot efficiency:

$$\epsilon = \frac{T_h - T_c}{T_h}, \tag{27}$$

which is often simplified as

$$\epsilon = 1 - \frac{T_c}{T_h}. \tag{28}$$